Stronger security bounds for permutations

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Abstract. It is well known that, inside any cryptographic protocol, a uniform random function can be safely replaced with a uniform random injective function: as long as the number of function queries is small, the attacker's success probability does not noticeably increase. This paper presents a quantitatively stronger theorem that handles a larger number of function queries. This quantitative improvement can be viewed as a generalization of the author's recent improvement in security bounds for Wegman-Carter-Shoup authenticators.

1 Introduction

Let p be a uniform random permutation of $S = \{0, 1, \ldots, 255\}^{16}$, and let f be a uniform random function from S to S. A cryptographic protocol using p is almost as hard to break as the same cryptographic protocol using f: it is well known that the difference in attacker success probabilities is at most $n(n-1)/2^{129}$ if there are n function queries.

Section 2 of this paper presents a quantitatively stronger theorem: if the attacker's success probability against f is at most ϵ then the attacker's success probability against p is at most $(1 - (n-1)/2^{128})^{-n/2}\epsilon$. For example, if $n = 2^{60}$ and $\epsilon = 1/2^{70}$, then the standard bound says that the attacker's success probability against p is at most $1/2^{70} + 2^{60}(2^{60} - 1)/2^{129} \approx 1/2^9$, while my bound says that it is at most $1/(1 - (2^{60} - 1)/2^{128})^{2^{59}} 2^{70} \approx 1/2^{70} + 1/2^{79}$.

As an illustration of the stronger theorem, this paper gives a new proof of my recent security bounds for Wegman-Carter-Shoup authenticators. The new proof can be viewed as a factorization of the original proof into (1) the main theorem of this paper and (2) the usual intuitive security bound for Wegman-Carter authenticators. See Sections 3 through 5.

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In a separate paper, as another illustration of the stronger theorem, I prove new security bounds for counter-mode encryption using p. I also show that CBC encryption is far below the same security level.

Application to short-key systems

Consider a cryptographic protocol using AES_k, where k is a uniform random 16-byte sequence. The attacker's success chance against this protocol is at most the sum of (1) the attacker's success chance against the same protocol using p and (2) the attacker's chance of distinguishing AES_k from p.

It is reasonable to conjecture that AES_k is hard to distinguish from p, even after a very large number of queries. This was an explicit design goal for AES; see [1, Section 4]. If the conjecture is true then this paper's improved security bounds for p imply improved security bounds for AES_k.

It is *not* reasonable to conjecture that AES_k is hard to distinguish from f. A simple n-query collision test distinguishes AES_k from f with probability approximately $n(n-1)/2^{129}$. This is what motivates studying p rather than f.

Previous work

As mentioned above, it is well known that the attacker's success probability against p is at most $\epsilon + n(n-1)/2^{129}$. This bound can be found in, e.g., [3, Section 2] and [8, Section 5.2].

Shoup in [7] proved better bounds for p in the specific context of Wegman-Carter authentication. I proved even better bounds in [4]. This paper generalizes those bounds to all protocols using p.

2 Main theorem

Theorem 2.3 is the permutation theorem announced in Section 1. Theorem 2.2 is a more general theorem, replacing a uniform random permutation with any random function having small interpolation probabilities. See [4, Section 4] for background on interpolation probabilities.

Theorem 2.1. *Let p be a random function from a set S to a nonempty finite set T. Let f be a uniform random function from S to T. Let n be an integer with $0 \le n \le \#S$. Let A be an algorithm that performs exactly n distinct oracle queries. Let δ be a positive real number. Assume that p has maximum n-interpolation probability at most $\delta/\#T^n$. Then* $\Pr[A(p) = 1] \le \delta \Pr[A(f) = 1]$.

Here $\Pr[A(p) = 1]$ means the probability that A prints 1 using p as an oracle; similarly for $\Pr[A(f) = 1]$. The hypothesis on interpolation probabilities is that $\Pr[(p(s_1), p(s_2), \ldots, p(s_n)) = (t_1, t_2, \ldots, t_n)] \le \delta/\#T^n$ for all $(t_1, t_2, \ldots, t_n) \in T^n$ and all $(s_1, s_2, \ldots, s_n) \in S^n$ with $s_1, s_2, \ldots, s_n$ distinct.

Proof. In a nutshell: Write $\alpha(t_1, t_2, \ldots, t_n)$ for the conditional probability that A prints 1 given that the responses to A's distinct oracle queries are $t_1, t_2, \ldots, t_n$. Then $\Pr[A(p) = 1] \le \sum_{t \in T^n} \alpha(t)\delta/\#T^n = \delta \sum_t \alpha(t)/\#T^n = \delta \Pr[A(f) = 1]$.

What follows is a more detailed version of the same proof.

Everything that A does is determined by (1) an infinite sequence b of coin flips, by definition independent of A's input, and (2) the responses $r_1, r_2, \ldots, r_n$ to A's distinct oracle queries. In particular, A's first oracle query is $q_1(b)$ for some function q_1; its second distinct oracle query is $q_2(b, r_1)$ for some function q_2; etc.; its final distinct oracle query is $q_n(b, r_1, r_2, \ldots, r_{n-1})$ for some function q_n; and its output is $z(b, r_1, r_2, \ldots, r_n)$ for some function z.

Fix $(t_1, t_2, \ldots, t_n) \in T^n$. The set $\{b : z(b, t_1, t_2, \ldots, t_n) = 1\}$ is recognizable and thus measurable; define $\alpha(t_1, t_2, \ldots, t_n)$ as its measure. Then $\alpha(t_1, t_2, \ldots, t_n)$ is exactly the conditional probability that A prints 1, given that $(r_1, r_2, \ldots, r_n) = (t_1, t_2, \ldots, t_n)$.

Consider A using p as an oracle. The probability that $(r_1, r_2, \ldots, r_n) = (t_1, t_2, \ldots, t_n)$ is exactly the probability that

$$(p(q_1(b)), p(q_2(b, t_1)), \ldots, p(q_n(b, t_1, t_2, \ldots, t_{n-1}))) = (t_1, t_2, \ldots, t_n).$$

The inputs $q_1(b), q_2(b, t_1), \ldots$ are independent of p, so this probability is at most the maximum n-interpolation probability of p, which by hypothesis is at most $\delta/\#T^n$.

Hence the probability that $(r_1, r_2, \ldots, r_n) = (t_1, t_2, \ldots, t_n)$ *and* A prints 1 is at most $\alpha(t_1, t_2, \ldots, t_n)\delta/\#T^n$. Sum over all choices of $(t_1, t_2, \ldots, t_n)$ to see that $\Pr[A(p) = 1] \le \sum_{t \in T^n} \alpha(t)\delta/\#T^n$.

Similarly, when A uses f as an oracle, the probability that $(r_1, r_2, \ldots, r_n) = (t_1, t_2, \ldots, t_n)$ *and* A prints 1 is exactly $\alpha(t_1, t_2, \ldots, t_n)/\#T^n$. Sum over all choices of $(t_1, t_2, \ldots, t_n)$ to see that $\Pr[A(f) = 1] = \sum_{t \in T^n} \alpha(t)/\#T^n$.

Hence $\Pr[A(p) = 1] \le \delta \Pr[A(f) = 1]$. □

Theorem 2.2. *Let p be a random function from a set S to a nonempty finite set T. Let f be a uniform random function from S to T. Let n be an integer with $0 \le n \le \#S$. Let A be an algorithm that performs at most n distinct oracle queries. Let δ be a positive real number. Assume that p has maximum n-interpolation probability at most $\delta/\#T^n$. Then* $\Pr[A(p) = 1] \le \delta \Pr[A(f) = 1]$.

Proof. Modify A into an algorithm A' that produces the same results but always performs exactly n distinct oracle queries. This means caching the oracle queries and—just before finishing—performing more queries if necessary, throwing away the results, to raise the total number of distinct queries to n. Note that new elements of S are available since $n \le \#S$.

Now $\Pr[A(p) = 1] = \Pr[A'(p) = 1] \le \delta \Pr[A'(f) = 1] = \delta \Pr[A(f) = 1]$ by Theorem 2.1. □

Theorem 2.3. *Let p be a uniform random permutation of a nonempty finite set S. Let f be a uniform random function from S to S. Let n be an integer with $0 \le n \le \#S$. Let A be an algorithm that performs at most n distinct oracle queries. Then* $\Pr[A(p) = 1] \le (1 - (n-1)/\#S)^{-n/2} \Pr[A(f) = 1]$.

Proof. Write $\delta = (1 - (n-1)/\#S)^{-n/2}$. Then p has maximum n-interpolation probability at most $\delta/\#S^n$ by [4, Theorem 4.2]. Apply Theorem 2.2 with $T = S$: $\Pr[A(p) = 1] \le \delta \Pr[A(f) = 1]$. □

Comparison to the standard bound

Substitute $(1-(n-1)/\#S)^{n/2} \ge 1-n(n-1)/2\#S$ into Theorem 2.3 to see that $\Pr[A(f) = 1] \ge \Pr[A(p) = 1] - (n(n-1)/2\#S)\Pr[A(p) = 1]$. Then substitute $\Pr[A(p) = 1] \le 1$ to see that $\Pr[A(f) = 1] \ge \Pr[A(p) = 1] - n(n-1)/2\#S$, i.e., $\Pr[A(p) = 1] \le \Pr[A(f) = 1] + n(n-1)/2\#S$. This is exactly the standard bound.

The first substitution is unnoticeable when n is small. Even for n as large as $\sqrt{\#S}$, the first substitution does not lose very much: $(1-(n-1)/\#S)^{n/2} \approx \exp(-1/2) \approx 0.6$ while $1 - n(n-1)/2\#S \approx 0.5$. But the second substitution is awful whenever $\Pr[A(f) = 1]$ is small: for example, it changes $\Pr[A(p) = 1] \le 2\Pr[A(f) = 1]$ into $\Pr[A(p) = 1] \le \Pr[A(f) = 1] + 0.5$.

I am not aware of a simpler proof of the standard bound. (The proofs in [3, Section 2] and [8, Section 5.1] are buried under a thicket of "game-playing" notation. The partial proof in [3, Appendix A], correcting the folklore "proof" criticized in [3, Section 2], follows the same strategy as Theorem 2.1.) In other words, the new bound is not only quantitatively better, but it is easier to prove: simply skip the extra substitutions!

3 MAC application, step 1: handle the uniform-random-function case

The Wegman-Carter message-authentication protocol uses, among other things, a uniform random function f. This section reviews the Wegman-Carter protocol and proves the usual security bound for the protocol against a single forgery attempt.

The next section will illustrate Theorem 2.3 by switching from a uniform random function f to a uniform random permutation p. Section 5 will generalize to a variable number of forgery attempts.

The final theorem is identical to my recent security bound [4, Theorem 5.3]. See [4] for further background. What is new in this paper is a factorization of the proof, one factor being Theorem 2.3 and another factor being the simpler security bound in this section.

The Wegman-Carter protocol

The Wegman-Carter protocol has several parameters: a finite commutative group G of **authenticators**; a nonempty set M of **messages**; and a finite set N of **nonces** with $\#N \le \#G$. The protocol has two independent secrets: a random function h from M to G, and a uniform random function f from N to G. The protocol has several participants:

- a **message generator** creates messages;
- a **nonce generator** attaches a nonce n to each message m, never using the same nonce for two different messages;
- a **sender** converts each pair (n, m) into $(n, m, h(m) + f(n))$;
- a **network** accepts a sequence of vectors (n, m, a) and transmits a sequence of vectors (n', m', a');
- a **receiver** receives vectors (n', m', a') from the network and accepts them if $a' = h(m') + f(n')$.

The secrets are shared by the sender and receiver.

The objective of the protocol is **forgery elimination**: ensuring that each (n', m', a') accepted by the receiver is one of the vectors (n, m, a) produced by the sender.

This protocol is usually stated in much less generality. In particular, Wegman and Carter did not actually consider group operations other than exclusive-or—but state-of-the-art choices of h require other group operations. See [4, Sections 1–2] for further background.

Attacks against the Wegman-Carter protocol

The combined behavior of the message generator, nonce generator, and network is called an "attack." The attack can access h and f only through the limited functions computed by the sender and receiver.

Formally: An **attack** is an algorithm given two oracles S, R as input. The algorithm feeds the first oracle any number of **chosen messages** (n, m), obeying the rule that distinct messages have distinct nonces. Meanwhile, the algorithm feeds the second oracle any number of **forgery attempts** (n', m', a'). The attack **succeeds against** S **and** R if at least one forgery attempt (n', m', a') has $R(n', m', a') = 1$ with (n', m') different from the previous queries to the first oracle.

Note that attackers are presumed to have control over messages; attackers can try to gain information from the sender and receiver; and forgery attempts are not required to be chosen in advance. See [4, Section 3] for further discussion of the definition of an attack.

Theorem 3.1. *Let h be a random function from a nonempty set M to a finite commutative group G. Let f be a uniform random function from a finite set N to G. Let ϵ be a real number with $\epsilon \geq 1/\#G$. Assume, for all $g \in G$ and all distinct $m, m' \in M$, that $h(m) = h(m') + g$ with probability at most ϵ. Assume that h and f are independent. Then any attack that performs exactly one forgery attempt succeeds against $(n, m) \mapsto h(m) + f(n)$ and $(n, m, a) \mapsto [a = h(m) + f(n)]$ with probability at most ϵ.*

The following proof is standard, although it is normally stated in much less generality.

Proof. For each choice of h, the sequence of responses from the first oracle has the same distribution, namely the uniform distribution. Indeed, each response is $h(m) + f(n)$ for a unique n; the values $f(n)$ are independent uniform random elements of G.

Consequently, the responses from the first oracle provide no information about h: the conditional distribution of h, given the responses, is identical to the original distribution of h. In particular, $h(m) = h(m') + g$ with conditional probability at most ϵ.

I now claim, for each possible sequence of previous responses from the first oracle and each possible sequence of coin flips in the attack, that the resulting forgery attempt (n', m', a') succeeds with conditional probability at most ϵ.

Case 1: (n', m') was a previous query to the first oracle. Then the forgery attempt is successful with conditional probability 0.

Case 2: (n', m) was a previous query to the first oracle for some $m \neq m'$. Write a for the oracle response; i.e., $a = h(m) + f(n')$. Then the forgery attempt is successful if and only if $h(m') - h(m) = a' - a$; this occurs with conditional probability at most ϵ.

Case 3: The nonce n' was not used in previous queries to the first oracle. Then $f(n')$ is independent of the responses, so the conditional distribution of $a' - h(m') - f(n')$ is uniform. The forgery attempt succeeds with conditional probability $1/\#G \leq \epsilon$. □

4 MAC application, step 2: apply the main theorem

Here is a generalization of the Wegman-Carter authentication protocol discussed in the previous section: replace the uniform random function f with any random function having small interpolation probabilities. For example, the Wegman-Carter-Shoup authentication protocol takes $N = G$ and replaces the uniform random function f with a uniform random permutation p.

This section uses Theorem 2.2 to deduce security bounds for the general case from security bounds for the uniform random case.

Theorem 4.1. *Let h be a random function from a nonempty set M to a finite commutative group G. Let p be a random function from a finite set N to G. Let C be a positive integer with $C + 1 \leq \#N$. Let δ be a positive real number. Let ϵ be a real number with $\epsilon \geq 1/\#G$. Assume that p has maximum $(C + 1)$-interpolation probability at most $\delta/\#T^{C+1}$. Assume, for all $g \in G$ and all distinct $m, m' \in M$, that $h(m) = h(m') + g$ with probability at most ϵ. Assume that h and p are independent. Then any attack that uses at most C distinct chosen messages and exactly one forgery attempt succeeds against $(n, m) \mapsto h(m) + p(n)$ and $(n, m, a) \mapsto [a = h(m) + p(n)]$ with probability at most $\delta\epsilon$.*

In particular, consider the special case that p is a uniform random injective function from N to G. Then p has maximum $(C+1)$-interpolation probability at most $(1 - C/\#G)^{-(C+1)/2}/\#T^{C+1}$ by [4, Theorem 4.2], so the forgery attempt succeeds with probability at most $(1 - C/\#G)^{-(C+1)/2}\epsilon$.

Proof. Convert the random function h into a random algorithm A that, given as input an oracle for a function z, carries out the attack against $(n, m) \mapsto h(m) + z(n)$ and $(n, m, a) \mapsto [a = h(m) + z(n)]$, and prints 1 if the attack succeeds. Note that A performs at most $C + 1$ distinct queries to z: one query for each distinct chosen message and one query for the forgery attempt.

Let f be a uniform random function from N to G, independent of h and p. Then f and p are independent of A. Apply Theorem 2.2 with $S = N$, $T = G$, and $n = C + 1$ to see that $\Pr[A(p) = 1] \leq \delta \Pr[A(f) = 1]$.

Now $\Pr[A(f) = 1]$ is exactly the probability that the attack succeeds against $(n, m) \mapsto h(m)+f(n)$ and $(n, m, a) \mapsto [a = h(m)+f(n)]$, so $\Pr[A(f) = 1] \leq \epsilon$ by Theorem 3.1. Similarly, $\Pr[A(p) = 1]$ is exactly the probability that the attack succeeds against $(n, m) \mapsto h(m)+p(n)$ and $(n, m, a) \mapsto [a = h(m)+p(n)]$. Hence the latter probability is at most $\delta\epsilon$ as claimed. □

The algorithms A considered in this proof are restricted by the structure of the authentication protocol and by the meaning of a successful attack. These restrictions produce bounds on $\Pr[A(f) = 1]$; Theorem 2.2 then produces bounds on $\Pr[A(p) = 1]$. Other cryptographic protocols put other constraints on A, and thus require different proofs that $\Pr[A(f) = 1]$ is small; but Theorem 2.2 then applies in exactly the same way to show that $\Pr[A(p) = 1]$ is small.

5 MAC application, step 3: allow more forgery attempts

Theorem 4.1 states that an attack against $h(m)+p(n)$ succeeds with probability at most $\delta\epsilon$ if it tries exactly one forgery attempt. This section generalizes to any number of forgery attempts: Theorem 5.2 states that an attack succeeds with probability at most $D\delta\epsilon$ if it tries at most D forgery attempts.

This has nothing to do with the particular shape $h(m) + p(n)$. Theorem 5.1 switches from single-forgery attacks to multiple-forgery attacks for arbitrary message-authentication codes $(n, m) \mapsto S(n, m)$.

It is important to perform the switch of the last section, generalizing from f to p, before the switch of this section, generalizing from single-forgery attacks to multiple-forgery attacks. There are at most $C + 1$ invocations of f in a C-chosen-message single-forgery attack, whereas there could be as many as $C + D$ invocations of f in a C-chosen-message D-forgery attack.

One could reduce both $C+1$ and $C+D$ to C, and thus allow the switches to be performed in either order, by modifying the definition of an attack, requiring all forgery attempts to repeat nonces that were used for chosen messages. But I prefer to keep the definition of an attack as broad as possible. A proof of security against restricted attacks forces the user to worry that the restriction artificially increases the protocol's security level (as this restriction does for $\#M = 1$). To the extent that security against restricted attacks implies security against all attacks, the cryptographer should take advantage of that to prove security against all attacks, rather than shifting the burden of proof to the user.

Theorem 5.1. *Let M be a nonempty set. Let N be a finite set. Let S be a random function from $N \times M$ to a finite commutative group G. Let C and D be positive integers. Let ϵ be a positive real number. Assume that any attack that uses at most C distinct chosen messages and exactly one forgery attempt succeeds against $(n, m) \mapsto S(n, m)$ and $(n, m, a) \mapsto [a = S(n, m)]$ with probability at most ϵ. Then any attack that uses at most C distinct chosen messages and at most D forgery attempts succeeds against $(n, m) \mapsto S(n, m)$ and $(n, m, a) \mapsto [a = S(n, m)]$ with probability at most $D\epsilon$.*

There is nothing special here about the particular restriction "at most C distinct chosen messages," but that restriction is natural whenever Theorem 5.1 is combined with Theorem 2.2.

As pointed out by Bellare, Goldreich, and Mityagin in [2], the same switch does not work for authentication protocols in which the receiver accepts several authenticators for a single (n, m). Each unsuccessful forgery attempt can leak as much as one bit of information, perhaps doubling the success probability of subsequent forgery attempts.

The crucial point in Theorem 5.1 is that the attacker can recognize all (n', m', a') that will be accepted by the receiver without being forgeries: namely, the results already obtained from the sender.

The theorems in [2] illustrate the same point but are not general enough. [2, Theorem 5.1] does not allow nonces; [2, Proposition 6.3] allows nonces but is limited to the Wegman-Carter protocol. The proof here is slightly more direct than the proof of [2, Theorem 5.1] and much more general than the proof of [2, Proposition 6.3].

Proof. Induct on D. For $D = 1$, there is nothing to prove, so assume that $D \geq 2$.

Let A be an attack that uses at most C distinct chosen messages and at most D forgery attempts. There are two ways for A to succeed: (1) it succeeds on its first forgery attempt; (2) it fails on its first forgery attempt but succeeds on a subsequent forgery attempt. I will show that the first way occurs with probability at most ϵ, and the second way occurs with probability at most $(D-1)\epsilon$, for a total of $D\epsilon$ as claimed.

(1) Define A_1 as A with one change: A_1 stops immediately after the first forgery attempt (if there are any forgery attempts).

A_1 uses at most C distinct chosen messages and at most 1 forgery attempt, so it succeeds with probability at most ϵ. Success of A on its first forgery attempt is equivalent to success of A_1, and therefore occurs with probability at most ϵ.

(2) Define A_2 as A with one change: A_2 simulates the first forgery attempt internally (if there are any forgery attempts) rather than sending the forgery attempt as an oracle query. The simulator returns 1 if the forgery attempt (n', m', a') matches an authenticator a' already provided in response to a chosen message (n', m'); otherwise the simulator returns 0.

A_2 uses at most C distinct chosen messages and at most $D-1$ forgery attempts, so by the inductive hypothesis it succeeds with probability at most $(D-1)\epsilon$. Failure of A on its first forgery attempt, combined with success on a subsequent attempt, implies success of A_2, and therefore occurs with probability at most $(D-1)\epsilon$. $\square$

Theorem 5.2. *Let h be a random function from a nonempty set M to a finite commutative group G. Let p be a random function from a finite set N to G. Let C and D be positive integers. Let δ be a positive real number. Let ϵ be a real number with $\epsilon \geq 1/\#G$. Assume that $C+1 \leq \#N$. Assume that p has maximum $(C+1)$-interpolation probability at most $\delta/\#T^{C+1}$. Assume, for all $g \in G$ and all distinct $m, m' \in M$, that $h(m) = h(m') + g$ with probability at most ϵ. Assume that h and p are independent. Then any attack that uses at most C distinct chosen messages and at most D forgery attempts succeeds against $(n, m) \mapsto h(m) + p(n)$ and $(n, m, a) \mapsto [a = h(m) + p(n)]$ with probability at most $D\delta\epsilon$.*

In particular, if p is a uniform random injective function from N to G, then the attack succeeds with probability at most $D(1-C/\#G)^{-(C+1)/2}\epsilon$. This special case is exactly [4, Theorem 5.3]. What is new here is the proof, factoring my previous proof into

- Theorem 3.1, the Wegman-Carter security bound;
- Theorem 2.2, the switch from a uniform random function f to a uniform random injective function p; and
- Theorem 5.1, the switch from single-forgery attacks to D-forgery attacks.

The third factor was reasonably explicit in my previous proof, but the first and second factors were not visible.

For general p, Theorem 5.2 is almost exactly [4, Theorem 5.1], the main theorem of [4]. The only difference is that Theorem 5.2 has a slightly simpler (and conceivably slightly stronger) interpolation-probability hypothesis.

Proof. Define $S(n, m) = h(m) + p(n)$.

By Theorem 4.1, any attack that uses at most C distinct chosen messages and exactly one forgery attempt succeeds against $(n, m) \mapsto S(n, m)$ and $(n, m, a) \mapsto [a = S(n, m)]$ with probability at most $\delta\epsilon$.

By Theorem 5.1, any attack that uses at most C distinct chosen messages and at most D forgery attempts succeeds against $(n, m) \mapsto S(n, m)$ and $(n, m, a) \mapsto [a = S(n, m)]$ with probability at most $D\delta\epsilon$. $\square$

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