PRIME SIEVES USING BINARY QUADRATIC FORMS

A. O. L. ATKIN AND D. J. BERNSTEIN

ABSTRACT. We introduce an algorithm that computes the prime numbers up to N using $O(N/\log\log N)$ additions and $N^{1/2+o(1)}$ bits of memory. The algorithm enumerates representations of integers by certain binary quadratic forms. We present implementation results for this algorithm and one of the best previous algorithms.

1. INTRODUCTION

Pritchard in [14] asked whether it is possible to print the prime numbers up to N, in order, using $o(N)$ operations and $O(N^\alpha)$ bits of memory for some $\alpha < 1$. Here "memory" does not include the paper used by the printer. "Operations" refers to loads, stores, comparisons, additions, and subtractions of $O(\log N)$-bit integers.

The answer is yes. We present a new algorithm that uses $o(N)$ operations and $N^{1/2+o(1)}$ bits of memory. We also present some implementation results; the new method is useful in practice.

This paper is not the end of the story. Galway in [8] and [9] started from the method described here and replaced certain subroutines, namely the algorithms in Section 4 of this paper, with computational versions of Sierpinski's theorem on the circle problem. The resulting algorithm uses somewhat more operations but only $N^{1/3+o(1)}$ bits of memory.

Strategy. The idea of the sieve of Eratosthenes is to enumerate values of the reducible binary quadratic form xy. The idea of the new algorithm is to enumerate values of certain *irreducible* binary quadratic forms. For example, a squarefree positive integer $p \in 1 + 4\mathbf{Z}$ is prime if and only if the equation $4x^2 + y^2 = p$ has an odd number of positive solutions (x, y). There are only $O(N)$ pairs (x, y) such that $4x^2 + y^2 \le N$.

We cover all primes $p > 3$ as follows. For $p \in 1 + 4\mathbf{Z}$ we use $4x^2 + y^2$ with $x > 0$ and $y > 0$; for $p \in 7 + 12\mathbf{Z}$ we use $3x^2 + y^2$ with $x > 0$ and $y > 0$; for $p \in 11 + 12\mathbf{Z}$ we use $3x^2 - y^2$ with $x > y > 0$. Section 6 reviews the relevant facts about these quadratic forms.

(One can vary the forms and the (x, y) conditions. For example, for $p \in 1 + 4\mathbf{Z}$ one could use $x^2 + y^2$ with $x > y > 0$; for $p \in 3 + 8\mathbf{Z}$ one could use $2x^2 + y^2$ with $x > 0$ and $y > 0$; for $p \in 7 + 8\mathbf{Z}$ one could use $2x^2 - y^2$ with $x > y > 0$. There are many possible choices; we have not determined the optimal set of forms.)

A standard improvement in the sieve of Eratosthenes is to enumerate values of xy not divisible by 2, 3, or 5; see Section 2 for details. This reduces the number of

Date: 20020329.

1991 *Mathematics Subject Classification.* Primary 11Y11; Secondary 11E25.

The second author was supported by the National Science Foundation under grant DMS–9600083.

pairs (x, y) by a constant factor. Similarly, we enumerate values of our quadratic forms not divisible by 5; see Section 3 for details.

More generally, one can select an integer W and enumerate values relatively prime to W. One can save a factor of $\log\log N$ in the running time of the sieve of Eratosthenes by letting W grow slowly with N. The same is true of the new method. In Section 5 we show that one can enumerate the primes up to N using $O(N/\log\log N)$ operations and $N^{1/2+o(1)}$ bits of memory.

2. The sieve of Eratosthenes

The following algorithm is standard. It uses B bits of memory to compute the primes in an arithmetic progression of B numbers.

Algorithm 2.1. Given $d \in \{1, 7, 11, 13, 17, 19, 23, 29\}$, to print all primes of the form $30k + d$ with $L \leq k < L + B$:

1. Set $a_L \leftarrow 1$, $a_{L+1} \leftarrow 1$, $\ldots$, $a_{L+B-1} \leftarrow 1$.
2. For each prime $q \geq 7$ with $q^2 < 30L + 30B$:
3. For each k with $30k + d$ a nontrivial multiple of q:
4. Set $a_k \leftarrow 0$.
5. Print $30k + d$ for each k with $a_k = 1$.

"Nontrivial multiple of q" in step 3 means "mq for some $m > 1$" but can safely be replaced by "mq for some $m \geq q$."

One can run Algorithm 2.1 for each d, and merge the results, to find all the primes p with $30L \leq p < 30L + 30B$. This uses $8B$ bits of memory, not counting the space needed to store the set of primes q.

To enumerate the primes p in a larger interval, say $30L \leq p < 30L+60B$, one can enumerate first the primes between $30L$ and $30L + 30B$, then the primes between $30L + 30B$ and $30L + 60B$, reusing the same $8B$ bits of memory.

The number of iterations of step 4 of Algorithm 2.1 is approximately $B/7$ for $q = 7$, $B/11$ for $q = 11$, and so on. By Mertens's theorem, the sum $B\sum_q(1/q)$ is roughly $B(\log\log(30L + 30B) - 1.465)$. See [10, Theorem 427].

Implementation results. The second author's implementation of Algorithm 2.1, using the `gcc 2.8.1` compiler on an UltraSPARC-I/167, takes about $3.3 \cdot 10^9$ cycles to find the 50847534 primes up to 10^9. Here $B = 128128$; the UltraSPARC has 131072 bits of fast memory.

Notes. Singleton in [15] suggested chopping a large interval into small pieces and applying the sieve of Eratosthenes to each piece. The same idea was published independently in [4] and later in [2].

Sieving an arithmetic progression is the p-adic analogue of sieving a bounded interval. Presumably Eratosthenes did not bother writing down even numbers in his sieve.

Instead of running Algorithm 2.1 independently for each d, one can handle all d simultaneously for each q: find all nontrivial multiples of q between $30L$ and $30L + 30B$, and translate each multiple into a pair (k, d). See [12] for details. For sufficiently large q this saves time despite the added cost of translation.

One can include composite integers q in step 2 of Algorithm 2.1. For example, it is easy to run through all integers $q > 1$ with $q \bmod 30 \in \{1, 7, 11, 13, 17, 19, 23, 29\}$. This saves the space necessary to store the primes q, at a small cost in time.

3. Prime sieves using irreducible binary quadratic forms

The following algorithms are new. Each algorithm uses B bits of memory to compute primes in an arithmetic progression of B numbers. Algorithm 3.1 requires each number to be congruent to 1 modulo 4; Algorithm 3.2 requires each number to be congruent to 1 modulo 6; Algorithm 3.3 requires each number to be congruent to 11 modulo 12.

Algorithm 3.1. Given $d \in \{1, 13, 17, 29, 37, 41, 49, 53\}$, to print all primes of the form $60k + d$ with $L \le k < L + B$:

1. Set $a_L \leftarrow 0$, $a_{L+1} \leftarrow 0$, $\ldots$, $a_{L+B-1} \leftarrow 0$.
2. For each (x, y, k) with $x > 0$, $y > 0$, $L \le k < L+B$, and $4x^2+y^2 = 60k+d$:
3. Set $a_k \leftarrow 1 - a_k$.
4. For each prime $q \ge 7$ with $q^2 < 60L + 60B$:
5. For each k with $60k + d$ divisible by q^2:
6. Set $a_k \leftarrow 0$.
7. Print $60k + d$ for each k with $a_k = 1$.

Steps 2 and 3 count, for each k, the parity of the number of pairs (x, y) with $4x^2 + y^2 = 60k + d$. Theorem 6.1 says that $60k + d$ is prime if and only if the number of pairs is odd and $60k + d$ is squarefree. Steps 4, 5, and 6 eliminate each k for which $60k + d$ is not squarefree.

The condition $4x^2 + y^2 \in d + 60\mathbf{Z}$ in step 2 implies 16 possibilities (depending on d) for $(x \bmod 15, y \bmod 30)$. Each possibility can be handled by Algorithm 4.1 below. There are approximately $(4\pi/15)B$ iterations of step 3.

Algorithm 3.2. Given $d \in \{1, 7, 13, 19, 31, 37, 43, 49\}$, to print all primes of the form $60k + d$ with $L \le k < L + B$:

1. Set $a_L \leftarrow 0$, $a_{L+1} \leftarrow 0$, $\ldots$, $a_{L+B-1} \leftarrow 0$.
2. For each (x, y, k) with $x > 0$, $y > 0$, $L \le k < L+B$, and $3x^2+y^2 = 60k+d$:
3. Set $a_k \leftarrow 1 - a_k$.
4. For each prime $q \ge 7$ with $q^2 < 60L + 60B$:
5. For each k with $60k + d$ divisible by q^2:
6. Set $a_k \leftarrow 0$.
7. Print $60k + d$ for each k with $a_k = 1$.

Algorithm 3.2 is justified by Theorem 6.2. In step 2 there are 12 possibilities for $(x \bmod 10, y \bmod 30)$, each of which can be handled by Algorithm 4.2 below. There are approximately $(\pi\sqrt{0.12})B$ iterations of step 3.

Algorithm 3.3. Given $d \in \{11, 23, 47, 59\}$, to print all primes of the form $60k + d$ with $L \le k < L + B$:

1. Set $a_L \leftarrow 0$, $a_{L+1} \leftarrow 0$, $\ldots$, $a_{L+B-1} \leftarrow 0$.
2. For each (x, y, k) with $x > y > 0$, $L \le k < L + B$, and $3x^2 - y^2 = 60k + d$:
3. Set $a_k \leftarrow 1 - a_k$.
4. For each prime $q \ge 7$ with $q^2 < 60L + 60B$:
5. For each k with $60k + d$ divisible by q^2:
6. Set $a_k \leftarrow 0$.
7. Print $60k + d$ for each k with $a_k = 1$.

Algorithm 3.3 is justified by Theorem 6.3. In step 2 there are 24 possibilities for $(x \bmod 10, y \bmod 30)$, each of which can be handled by Algorithm 4.3 below. There are approximately $(\sqrt{1.92}\log(\sqrt{0.5} + \sqrt{1.5}))B$ iterations of step 3.

Implementation results. The second author's implementation of Algorithm 3.1, Algorithm 3.2, and Algorithm 3.3, using `gcc 2.8.1` on an UltraSPARC-I/167 with $B = 128128$, takes about $2.5 \cdot 10^9$ cycles to find the primes up to 10^9. For the code see `http://...2...2g.rg.ro......e.com/primegen.html`.

About 87% of the time was spent in steps 2 and 3 of these algorithms: 38% in Algorithm 3.1 for $d \in \{1, 13, 17, 29, 37, 41, 49, 53\}$; 26% in Algorithm 3.2 for $d \in \{7, 19, 31, 43\}$; 23% in Algorithm 3.3 for $d \in \{11, 23, 47, 59\}$. About half of the remaining time was spent in steps 4, 5, and 6.

Notes. One could change the "even, odd" counter a_k in Algorithm 3.1 to a "zero, one, more" counter, and then skip some values of q in step 4. The same comment applies to Algorithm 3.2 and Algorithm 3.3.

4. Enumerating lattice points

The idea of Algorithm 4.1 is to scan upwards from the lower boundary of the first quadrant of the annulus $60L \le 4x^2 + y^2 < 60L + 60B$. The total number of points considered by Algorithm 4.1 is $(1/450)(\pi/8)(60B) + O(\sqrt{60L + 60B})$. Here $(\pi/8)(60B)$ is the area of the quadrant, and $1/450$ accounts for the restriction on $(x \bmod 15, y \bmod 30)$. Similar comments apply to Algorithm 4.2 and Algorithm 4.3.

Algorithm 4.1. Given positive integers $d < 60$, $f \le 15$, and $g \le 30$ such that $d \equiv 4f^2 + g^2 \pmod{60}$, to print all triples (x, y, k) with $x > 0$, $y > 0$, $L \le k < L + B$, $4x^2 + y^2 = 60k + d$, $x \in f + 15\mathbf{Z}$, and $y \in g + 30\mathbf{Z}$:

1. Set $x \leftarrow f$, $y_0 \leftarrow g$, and $k_0 \leftarrow (4f^2 + g^2 - d)/60$. (Starting in step 3 we will move (x, y_0) along the lower boundary, from right to left, keeping track of $k_0 = (4x^2 + y_0^2 - d)/60$.)
2. If $k_0 < L + B$: Set $k_0 \leftarrow k_0 + 2x + 15$. Set $x \leftarrow x + 15$. Repeat this step.
3. (Move left.) Set $x \leftarrow x - 15$. Set $k_0 \leftarrow k_0 - 2x - 15$. Stop if $x \le 0$.
4. (Move up if necessary.) If $k_0 < L$: Set $k_0 \leftarrow k_0 + y_0 + 15$. Set $y_0 \leftarrow y_0 + 30$. Repeat this step.
5. (Now $4x^2 + y_0^2 \ge 60L$; and if $y_0 > 30$ then $4x^2 + (y_0 - 30)^2 < 60L$.) Set $k \leftarrow k_0$ and $y \leftarrow y_0$.
6. (Now $4x^2 + y^2 = 60k + d \ge 60L$.) If $k < L + B$: Print (x, y, k). Set $k \leftarrow k + y + 15$. Set $y \leftarrow y + 30$. Repeat this step.
7. Go back to step 3.

Algorithm 4.2. Given positive integers $d < 60$, $f \le 10$, and $g \le 30$ such that $d \equiv 3f^2 + g^2 \pmod{60}$, to print all triples (x, y, k) with $x > 0$, $y > 0$, $L \le k < L + B$, $3x^2 + y^2 = 60k + d$, $x \in f + 10\mathbf{Z}$, and $y \in g + 30\mathbf{Z}$:

1. Set $x \leftarrow f$, $y_0 \leftarrow g$, and $k_0 \leftarrow (3f^2 + g^2 - d)/60$.
2. If $k_0 < L + B$: Set $k_0 \leftarrow k_0 + x + 5$. Set $x \leftarrow x + 10$. Repeat this step.
3. Set $x \leftarrow x - 10$. Set $k_0 \leftarrow k_0 - x - 5$. Stop if $x \le 0$.
4. If $k_0 < L$: Set $k_0 \leftarrow k_0 + y_0 + 15$. Set $y_0 \leftarrow y_0 + 30$. Repeat this step.
5. Set $k \leftarrow k_0$ and $y \leftarrow y_0$.
6. If $k < L + B$: Print (x, y, k). Set $k \leftarrow k + y + 15$. Set $y \leftarrow y + 30$. Repeat this step.
7. Go back to step 3.

Algorithm 4.3. Given positive integers $d < 60$, $f \le 10$, and $g \le 30$ such that $d \equiv 3f^2 - g^2 \pmod{60}$, to print all triples (x, y, k) with $x > y > 0$, $L \le k < L + B$, $3x^2 - y^2 = 60k + d$, $x \in f + 10\mathbf{Z}$, and $y \in g + 30\mathbf{Z}$:

1. Set $x \leftarrow f$, $y_0 \leftarrow g$, and $k_0 \leftarrow (3f^2 - g^2 - d)/60$.
2. If $k_0 \ge L + B$: Stop if $x \le y_0$. Set $k_0 \leftarrow k_0 - y_0 - 15$. Set $y_0 \leftarrow y_0 + 30$. Repeat this step.
3. Set $k \leftarrow k_0$ and $y \leftarrow y_0$.
4. If $k \ge L$ and $y < x$: Print (x, y, k). Set $k \leftarrow k - y - 15$. Set $y \leftarrow y + 30$. Repeat this step.
5. Set $k_0 \leftarrow k_0 + x + 5$. Set $x \leftarrow x + 10$. Go back to step 2.

Notes. Tracing a level curve is a standard technique in computer graphics; see, e.g., [1, Chapter 17]. It is often credited to [5] but it appeared earlier in [11, Section 3].

5. Asymptotic performance

For large N one can compute the primes up to N as follows.

Define W as 12 times the product of all the primes from 5 up to about $\sqrt{\log N}$. Note that W is in $N^{o(1)}$; it is roughly $\exp\sqrt{\log N}$.

Write φ_1 for the number of units modulo W. Then φ_1/W is in $O(1/\log\log N)$ by Mertens's theorem.

Write φ_2 for the number of pairs $(x \bmod W, y \bmod W)$ such that $4x^2 + y^2$ or $3x^2 + y^2$ or $3x^2 - y^2$ is a unit modulo W. Then φ_2/W^2 is in $O(1/\log\log N)$ by a standard generalization of Mertens's theorem.

Select an integer B close to $W\sqrt{N}$. The method described in this section uses $\varphi_1 B$ bits of memory, i.e., $N^{1/2+o(1)}$ bits. We leave it to the reader to investigate how much the memory consumption can be reduced without noticeably affecting the number of operations.

Here is how to compute the primes in $d + W\mathbf{Z}$ between WL and $WL + WB$, given a positive integer $L < N/W$ and given a unit d modulo W:

- Define $(a, b) = (4, 1)$ if $d \in 1 + 4\mathbf{Z}$; define $(a, b) = (3, 1)$ if $d \in 7 + 12\mathbf{Z}$; define $(a, b) = (3, -1)$ if $d \in 11 + 12\mathbf{Z}$.
- Find all possible $(x \bmod W, y \bmod W)$ given that $ax^2 + by^2 \in d + W\mathbf{Z}$. This can easily be done in $W^{O(1)}$ operations.
- For each possible $(x \bmod W, y \bmod W)$, enumerate all (x, y) with $WL \le ax^2 + by^2 < WL + WB$, as in Section 4, and toggle the appropriate bits in a B-bit array. The number of operations here is $O(B/W)$ for each possible $(x \bmod W, y \bmod W)$: the choice of B guarantees that $\sqrt{WL + WB}$ is in $O(B/W)$.
- Eliminate numbers that are not squarefree, as in Section 3, to determine the primes. The number of operations here is $O(B)$.

The total number of operations over all d, to compute all the primes between WL and $WL + WB$ using φ_1 separate B-bit arrays, is $W^{O(1)} + O(\varphi_2 B/W) + O(\varphi_1 B) = O(WB/\log\log N)$.

Consequently one can compute all the primes up to N using $O(N/\log\log N)$ operations and $N^{1/2+o(1)}$ bits of memory.

Notes. Pritchard in [14] pointed out that, by the method of Section 2, one can compute the primes up to N using $O(N)$ operations and $O(N^{1/2}(\log\log N)/\log N)$ bits of memory.

By a similar method one can compute the primes up to N using $O(N/\log\log N)$ operations and $N^{1+o(1)}$ bits of memory. Pritchard gave a proof in [12] and a simpler proof in [13]. Dunten, Jones, and Sorenson in [6] reduced the amount of memory by a logarithmic factor; of course, $N^{1+o(1)}/\log N$ is still $N^{1+o(1)}$.

The new method uses $O(N/\log\log N)$ operations with only $N^{1/2+o(1)}$ bits of memory. No previous method achieves both bounds simultaneously.

6. Quadratic forms

Theorem 6.1. *Let n be a squarefree positive integer with $n \in 1 + 4\mathbf{Z}$. Then n is prime if and only if $\#\{(x, y) : x > 0, y > 0, 4x^2 + y^2 = n\}$ is odd.*

The following proof uses the fact that the unit group $\mathbf{Z}[i]^*$ of the principal ideal domain $\mathbf{Z}[i]$, where $i = \sqrt{-1}$, is $\{1, -1, i, -i\}$. The idea is to find representatives in $\mathbf{Z}[i]$ for the semigroup $\mathbf{Z}[i]/\mathbf{Z}[i]^*$.

Proof. The statement is true for $n = 1$, so assume $n > 1$.

Define $S = \{(x, y) : y > 0, 4x^2 + y^2 = n\}$. Define T as the set of norm-n ideals in $\mathbf{Z}[i]$. For each $(x, y) \in S$ define $f(x, y) \in T$ as the ideal generated by $y + 2xi$.

Step 1: f is injective. Indeed, the other generators of the ideal generated by $y+2xi$ are $-y-2xi$, $-2x+yi$, and $2x-yi$, none of which are of the form $y'+2x'i$ with $y' > 0$.

Step 2: f is surjective. Indeed, take any $I \in T$. Select a generator $a + bi$ of I; then $a^2 + b^2 = n$. Note that $b \neq 0$ since n is squarefree. If a is even and $b > 0$ then $I = f(-a/2, b)$; if a is even and $b < 0$ then $I = f(a/2, -b)$; if a is odd and $a > 0$ then $I = f(b/2, a)$; if a is odd and $a < 0$ then $I = f(-b/2, -a)$.

Step 3: If n is prime then $\#T = 2$ so $\#\{(x, y) : x > 0, y > 0, 4x^2 + y^2 = n\} = (\#S)/2 = (\#T)/2 = 1$. Otherwise write $n = p_1p_2\cdots p_r$ where each p_k is prime. The number of norm-p_k ideals is even, so $\#T$ is divisible by 2^r, hence by 4; thus $\#\{(x, y) : x > 0, y > 0, 4x^2 + y^2 = n\} = (\#S)/2 = (\#T)/2$ is even. □

Theorem 6.2. *Let n be a squarefree positive integer with $n \in 1 + 6\mathbf{Z}$. Then n is prime if and only if $\#\{(x, y) : x > 0, y > 0, 3x^2 + y^2 = n\}$ is odd.*

The following proof uses the fact that the unit group of the principal ideal domain $\mathbf{Z}[\omega]$, where $\omega = (-1 + \sqrt{-3})/2$, is $\{1, \omega, \omega^2, -1, -\omega, -\omega^2\}$.

Proof. Assume $n > 1$. Define $S = \{(x, y) : y > 0, 3x^2 + y^2 = n\}$. Define T as the set of norm-n ideals in $\mathbf{Z}[\omega]$. For each $(x, y) \in S$ define $f(x, y) \in T$ as the ideal generated by $x + y + 2x\omega$. If n is prime then $\#T = 2$; otherwise $\#T$ is divisible by 4. By calculations similar to those in Theorem 6.1 the reader may verify that f is a bijection from S to T. □

Theorem 6.3. *Let n be a squarefree positive integer with $n \in 11 + 12\mathbf{Z}$. Then n is prime if and only if $\#\{(x, y) : x > y > 0, 3x^2 - y^2 = n\}$ is odd.*

The following proof uses the fact that the unit group $\mathbf{Z}[\gamma]^*$ of the principal ideal domain $\mathbf{Z}[\gamma]$, where $\gamma = \sqrt{3}$, is $\{\pm(2 + \gamma)^j : j \in \mathbf{Z}\}$.

Proof. Define $S = \{(x,y) : |x| > y > 0, 3x^2 - y^2 = n\}$. Define T as the set of norm-n ideals in $\mathbf{Z}[\gamma]$. For each $(x,y) \in S$ define $f(x,y) \in T$ as the ideal generated by $y + x\gamma$. As above it suffices to show that f is a bijection from S to T.

Define $L = \log(2+\gamma)$, and define a homomorphism $\operatorname{Log} : \mathbf{Q}[\gamma]^* \to \mathbf{R}^2$ by $\operatorname{Log}(a + b\gamma) = (\log|a+b\gamma|, \log|a-b\gamma|)$. Then $\operatorname{Log}\mathbf{Z}[\gamma]^* = (L,-L)\mathbf{Z}$. Note that if $|b| > a > 0$ then $|u - v| < L$ where $(u,v) = \operatorname{Log}(a+b\gamma)$; and if $|u-v| \le L$ then either $|a| \le |b|$ or $|a| \ge 3|b|$.

Injectivity: For $(x,y) \in S$ and $(x',y') \in S$ write $(u,v) = \operatorname{Log}(y + x\gamma)$ and $(u',v') = \operatorname{Log}(y' + x'\gamma)$. Then $|u-v| < L$ and $|u'-v'| < L$, so $|u - v - u' + v'| < 2L$. Now assume that $f(x,y) = f(x',y')$. Then $(u,v) - (u',v') \in (L,-L)\mathbf{Z}$, so $(u,v) = (u',v')$, so $(x',y') \in \{(x,y),(-x,-y)\}$; but y and y' are both positive, so $(x',y') = (x,y)$.

Surjectivity: Given a norm-n ideal I, pick a generator $a + b\gamma$ of I. Write $(u,v) = \operatorname{Log}(a+b\gamma)$. Select an integer j within $1/2$ of $(v-u)/2L$, and write $y + x\gamma = (a+b\gamma)(2+\gamma)^j$. Then $\operatorname{Log}(y+x\gamma) = (u+jL, v-jL)$, and $|(u+jL)-(v-jL)| \le L$, so $|y| \le |x|$ or $|y| \ge 3|x|$. But $n = \pm(3x^2 - y^2)$, and $n \in 11 + 12\mathbf{Z}$, so $n = 3x^2 - y^2$; in particular $3x^2 - y^2 > 0$ so $|y| \le |x|$. Also $|y| \ne 0$ and $|y| \ne |x|$ since n is squarefree. If $y > 0$ then $I = f(x,y)$; if $y < 0$ then $I = f(-x,-y)$. □

Notes. These theorems are standard. See, e.g., [16, Chapter 11]. We have included proofs for the sake of completeness.

The function Log in the proof of Theorem 6.3 is an example of Dirichlet's log map. See, e.g., [7, page 169].

The approximations $(4\pi/15)B$, $(\pi\sqrt{0.12})B$, and $(\sqrt{1.92}\log(\sqrt{0.5}+\sqrt{1.5}))B$ for the number of lattice points considered in Section 3 and Section 4 are examples of Dirichlet's analytic class-number formula. See, e.g., [7, pages 283–294], particularly [7, page 289].

References

[1] Michael Abrash, *Zen of graphics programming*, Coriolis Group, Scottsdale, Arizona, 1995. ISBN 188357708X.

[2] Carter Bays, Richard H. Hudson, *The segmented sieve of Eratosthenes and primes in arithmetic progressions to* 10^{12}, BIT **17** (1977), 121–127. ISSN 0006–3835. MR 56 #5405.

[3] Wieb Bosma (editor), *Algorithmic number theory: ANTS-IV*, Lecture Notes in Computer Science, 1838, Springer-Verlag, Berlin, 2000. ISBN 3–540–67695–3. MR 2002d:11002.

[4] Richard P. Brent, *The first occurrence of large gaps between successive primes*, Mathematics of Computation **27** (1973), 959–963. MR 48 #8360. Available from `http://..rbc.road..uae.comlab.ox.ac.uk/oucl/work/richard.brent/pub/pub019.html`.

[5] Jack Bresenham, *A linear algorithm for incremental digital display of circular arcs*, Communications of the ACM **20** (1977), 100–106. ISSN 0001–0782.

[6] Brian Dunten, Julie Jones, Jonathan Sorenson, *A space-efficient fast prime number sieve*, Information Processing Letters **59** (1996), 79–84. ISSN 0020–0190. MR 97g:11141.

[7] Albrecht Fröhlich, Martin J. Taylor, *Algebraic number theory*, Cambridge University Press, Cambridge, 1991. ISBN 0–521–36664–X. MR 94d:11078.

[8] William F. Galway, *Dissecting a sieve to cut its need for space*, in [3] (2000), 297–312.

[9] William F. Galway, *Analytic computation of the prime-counting function*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2001.

[10] G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*, 5th edition, Oxford University Press, 1979. ISBN 0–19–853170–2. MR 81i:10002.

[11] Herbert B. Keller, J. R. Swenson, *Experiments on the lattice problem of Gauss*, Mathematics of Computation **17** (1963), 223–230. MR 29 #3445.

[12] Paul Pritchard, *A sublinear additive sieve for finding prime numbers*, Communications of the ACM **24** (1981), 18–23. ISSN 0001–0782. MR 82c:10011.

[13] Paul Pritchard, *Explaining the wheel sieve*, Acta Informatica **17** (1982), 477–485. ISSN 0001–5903. MR 84g:10015.
[14] Paul Pritchard, *Fast compact prime number sieves (among others)*, Journal of Algorithms **4** (1983), 332–344. ISSN 0196–6774. MR 85h:11080.
[15] Richard C. Singleton, *Algorithm 357: an efficient prime number generator*, Communications of the ACM **12** (1969), 563–564. ISSN 0001–0782.
[16] J. V. Uspensky, Max A. Heaslet, *Elementary number theory*, McGraw-Hill, New York, 1939. MR 1,38d.

Department of Mathematics, Statistics, and Computer Science (M/C 249), The University of Illinois at Chicago, Chicago, IL 60607–7045
E-mail address: aolatkin@uic.edu

Department of Mathematics, Statistics, and Computer Science (M/C 249), The University of Illinois at Chicago, Chicago, IL 60607–7045
E-mail address: djb@pobox.com